

# THE DEFECT VARIANCE OF RANDOM SPHERICAL HARMONICS

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**ABSTRACT.** The defect of a function  $f : M \rightarrow \mathbb{R}$  is defined as the difference between the measure of the positive and negative regions. In this paper, we begin the analysis of the distribution of defect of random Gaussian spherical harmonics. By an easy argument, the defect is non-trivial only for even degree and the expected value always vanishes. Our principal result is obtaining the asymptotic shape of the defect variance, in the high frequency limit. As other geometric functionals of random eigenfunctions, the defect may be used as a tool to probe the statistical properties of spherical random fields, a topic of great interest for modern Cosmological data analysis.

## 1. INTRODUCTION

In recent years, a lot of interest has been drawn by the analysis of the geometric features of random eigenfunctions for the spherical Laplacian. More precisely, let us denote as usual by  $\{Y_{lm}(\cdot)\}_{m=-l,\dots,l}$ , for  $l = 1, 2, \dots$  the set of (real-valued) spherical harmonics, i.e. the array of real-valued functions satisfying

$$\Delta_{S^2} Y_{lm} = -l(l+1)Y_{lm},$$

where  $\Delta_{S^2}$  denotes the spherical Laplacian (see for instance [17], [19]). The random model we shall focus on is

$$(1) \quad f_l(x) = \frac{1}{\sqrt{2l+1}} \sum_{m=-l}^l a_{lm} Y_{lm}(x)$$

where the coefficients  $\{a_{lm}\}$  are independent standard Gaussian with zero-mean and unit variance.

The random field  $f_l$  is isotropic, meaning that for any  $k \in \mathbb{N}$  and  $x_1, \dots, x_k \in \mathcal{S}^2$ , the distribution of the random vector  $\{f_l(x_1), \dots, f_l(x_k)\}$  is invariant under rotations, i.e. equals to the distribution of  $\{f_l(g \cdot x_1), \dots, f_l(g \cdot x_k)\}$  for any rotation  $g \in SO(3)$  on the sphere. Also,  $f_l$  is centred Gaussian, with covariance function

$$(2) \quad r_l(x, y) := \mathbb{E}[f_l(x) \cdot f_l(y)] = P_l(\cos(d(x, y))),$$

where  $P_l$  are the usual Legendre polynomials defined by Rodrigues' formula

$$P_l(t) := \frac{1}{2^l l!} \frac{d^l}{dt^l} (t^2 - 1)^l$$

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and  $d(x, y)$  is the (spherical) geodesic distance between  $x$  and  $y$ . As well-known, Legendre polynomials are orthogonal w.r.t. the constant weight  $\omega(t) \equiv 1$  on  $[-1, 1]$ , see Section 5 for more discussion and details.

**1.1. Background.** Random eigenfunctions for the spherical Laplacian naturally emerge in a number of different physical contexts. A particularly active area is related to the analysis of isotropic spherical random fields on the sphere, as motivated for instance by the analysis of Cosmic Microwave Background radiation (CMB), see for instance [5, 6] and the references therein. Under these circumstances, eigenfunctions like  $f_l$  represent the (normalized) Fourier components of the field, i.e. the following orthogonal expansion holds, in the  $L^2$  sense:

$$f(x) = \sum_{l=1}^{\infty} c_l f_l(x) ,$$

where  $c_l \geq 0$  is a deterministic sequence (depending on the angular power spectrum) which encodes the full correlation structure of  $f$ .

In the CMB literature,  $f(x)$ , and hence the components  $f_l$ , are actually observed by highly sophisticated satellite experiments such as *WMAP* by NASA and *Planck* by ESA. It is then common practice to analyze geometric functionals of the observed CMB radiation to constraint the statistical properties of the underlying fields, e.g. to test for isotropy and/or Gaussianity. For instance, the three Minkowski functionals, providing the area, the boundary length and the topological genus of excursion sets over a given level ( $A_z := \{x \in \mathcal{S}^2 : f(x) \geq z\}$ ) have been applied on CMB data by a huge number of authors, including [8, 9, 13]. Many efforts have been spent to analyze unexpected features; for instance, the so-called Cold Spot (see [3, 4] and the references therein), whose statistical significance can also be evaluated by means of local curvature properties of isotropic Gaussian fields ([7]).

In this work, we shall focus on one of the most important geometric functionals, namely the *defect*. The defect (or “signed area”, see [2]) of a function  $\psi : \mathbb{S}^2 \rightarrow \mathbb{R}$  is defined as

$$(3) \quad \mathcal{D}(\psi) := \text{meas}(\psi^{-1}(0, \infty)) - \text{meas}(\psi^{-1}(-\infty, 0)) = \int_{\mathbb{S}^2} \mathcal{H}(\psi(x)) dx.$$

Here  $\mathcal{H}(t)$  is the Heaviside function

$$(4) \quad \mathcal{H}(t) = \mathbb{1}_{[0, \infty)}(t) - \mathbb{1}_{(-\infty, 0]}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \\ 0 & t = 0 \end{cases},$$

where  $\mathbb{1}_A(t)$  is the usual indicator function of the set  $A$ , and  $dx$  is the Lebesgue measure. The defect is hence the difference between the areas of positive and negative inverse image of  $\psi$ , respectively.

Of course, the defect  $\mathcal{D}_l = \mathcal{D}(f_l)$  of  $f_l$  is a random variable; its distribution is the primary focus of the present paper. Note that for odd  $l$ ,  $f_l$  is always odd, so that in this case the defect vanishes identically ( $\mathcal{D}_l \equiv 0$ ), and therefore  $\mathcal{D}_l$  has nontrivial distribution for even  $l$  only. A possible alternative to avoid trivialities is to restrict ourselves to a subset of the sphere, the

most natural choice being a hemisphere, i.e. we may choose any hemisphere  $\mathcal{E} \subseteq \mathcal{S}^2$  and define

$$\mathcal{D}_l^\mathcal{E} := \int_{\mathcal{E}} \mathcal{H}(f_l(x)) dx;$$

in this paper we study  $\mathcal{D}_l$  for even  $l$  only.

**1.2. Statement of the main result.** Evaluating the expectation of  $\mathcal{D}_l$  is trivial. Indeed, notice that integration over  $\mathcal{S}^2$  is exchangeable with expectation, so that

$$\mathbb{E}[\mathcal{D}_l] = \int_{\mathcal{S}^2} \mathbb{E}[\mathcal{H}(f_l(x))] dx,$$

and  $\mathbb{E}[\mathcal{H}(f_l(x))] = 0$  vanishes for every  $x \in \mathcal{S}^2$ , by the symmetry of the Gaussian distribution. We just established the following lemma:

**Lemma 1.1.** *For every  $l = 1, 2, \dots$ , we have*

$$\mathbb{E}[\mathcal{D}_l] = 0.$$

The main result of the present paper concerns the asymptotic behaviour of the defect variance:

**Theorem 1.2.** *As  $l \rightarrow \infty$  along even integers, the defect variance is asymptotic to*

$$(5) \quad \text{Var}(\mathcal{D}_l) = \frac{C}{l^2}(1 + o(1)),$$

where  $C > 0$  is a positive constant.

The constant  $C$  in (5) may be expressed in terms of the infinite (conditionally convergent) integral

$$(6) \quad C = 32\pi \int_0^\infty \psi (\arcsin(J_0(\psi)) - J_0(\psi)) d\psi.$$

See Section 2 for some details on the integral on the RHS of (6). We do not know whether one can evaluate  $C$  explicitly; however, we shall be able to show that

$$C > \frac{32}{\sqrt{27}},$$

(see Lemma 5.5).

One should compare the statement of Theorem 1.2 to empirical results of the study conducted by Blum, Gnuzmann and Smilansky [2]. The authors of that work studied the defect (or, as they refer to, the “signed area”) of random monochromatic waves on various planar domains. For the particular case of unit circle, they found that the order of magnitude of defect variance is consistent to (5) per unit area, with leading constant evaluated numerically as  $\approx 0.0386$ . The corresponding defect variance per unit area in our situation is

$$\text{Var}\left(\frac{\mathcal{D}_l}{4\pi}\right) = \frac{\tilde{C}}{l^2}(1 + o(1))$$

with leading constant

$$\tilde{C} > \frac{2}{\pi^2 \times \sqrt{27}} = 0.0389\dots$$

**1.3. Previous work.** To put our results in a proper perspective and explain the technical difficulties to be handled, we need to briefly recall some results from [12]. In that work we studied the asymptotic behaviour of excursion sets

$$A_z = f_l^{-1}([z, \infty))$$

of  $f_l$ , as  $l \rightarrow \infty$ . As we will recall below, a rather peculiar phenomenon can be shown to hold for  $z \neq 0$ ; namely, the asymptotic distribution of the area of these excursion sets is fully degenerate over  $z$ , i.e. it corresponds to a Gaussian random variable times a deterministic function of  $z$ . This result is intuitively due to an asymptotic degeneracy in the excursion set functionals, which turns out to be dominated by a single polynomial (quadratic) term.

This degeneracy, however, does not hold for the special case  $z = 0$ . More precisely, in [12] we focused on the asymptotic behaviour of the empirical measure of random spherical harmonics, defined as

$$\Phi_l(z) := \int_{S^2} \mathbb{1}_{[z, \infty)}(f_l(x)) dx = \text{meas} \{x : f_l(x) \leq z\},$$

$z \in \mathbb{R}$ . A key step in that paper is the asymptotic expansion

$$(7) \quad \Phi_l(z) = 4\pi \times \Phi(z) + \sum_{q=1}^{\infty} \frac{J_q(z)}{q!} h_{l;q}, \quad h_{l;q} := \int_{S^2} H_q(f_l(x)) dx,$$

where  $H_q(\cdot)$  are standard Hermite polynomials,  $\Phi(z) = \Pr\{Z \leq z\}$  is the cumulative distribution function of a standard Gaussian variable, and the deterministic functions  $J_q(z)$  can be explicitly provided in terms of higher order derivatives of  $\Phi(z)$ ,  $J_q(z) = (-1)^q \Phi^{(q)}(z)$  (see [12] for more discussion and details).

It turns out that, as  $l \rightarrow \infty$ ,

$$\frac{\sqrt{l}}{4\pi} h_{l;2} \rightarrow_d N(0, 1), \quad \sqrt{l} h_{l;q} = o_p(1), \quad \text{for } q \geq 3;$$

moreover  $J_2(z) = -z\phi(z)$  clearly does not vanish for all  $z \neq 0$ , whence the asymptotic behaviour of  $\sqrt{l} \{\Phi_l(z) - 4\pi \times \Phi(z)\}$  is easily seen to be Gaussian, uniformly over  $z$ . Furthermore, the limiting process is completely degenerate with respect to  $z$ , a feature to which we shall come back later.

For  $z \neq 0$ , the asymptotic behaviour of the area functional for the excursion sets is hence fully understood. The previous argument, however, fails for  $z = 0$ , as in this case the leading term is null and each summand in the asymptotic expansion (7) becomes relevant. Up to a linear transformation, this is clearly equivalent to the defect functional, indeed

$$\mathcal{D}_l = 4\pi - 2\Phi_l(0).$$

Thus, the case  $z = 0$  is the most challenging from the mathematical point of view, and, at the same time, the most interesting from the point of view of geometric interpretation.

**1.4. Overview of the paper.** The plan of the paper is as follows. In Section 2, we provide the main ideas behind our principal arguments, to help the reader understand the material to follow; in Section 3, we discuss the relation of our results to recent works on the distribution of nodal lengths

and level curves for random eigenfunctions and related conjectures; Section 4 provides the proof of the main results, whereas Section 5 contains auxiliary lemmas, we believe of independent interest, on the asymptotic behaviour of moments of Legendre polynomials.

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## 2. ON THE PROOF OF THEOREM 1.2

To establish our results, we shall need a detailed analysis of the odd moments of Legendre polynomials

$$\int_0^1 P_l(t)^{2k+1} dt = \int_0^{\pi/2} P_l(\cos \theta)^{2k+1} \sin \theta d\theta.$$

The rationale for this can be explained as follows. It is relatively easy to express (up to a constant) the defect variance as

$$(8) \quad I_l = \int_0^{\pi/2} \arcsin(P_l(\cos \theta)) \sin \theta d\theta,$$

where  $\sin \theta d\theta$  is (up to a constant) the uniform measure on the sphere in the spherical coordinates (see Lemma 4.1). It then remains to understand the asymptotic behaviour of  $I_l$ , a task which is put forward in Proposition 4.2; here we provide the main ideas underlying its proof. We know from Hilb's asymptotics (Lemma 5.1), that  $P_l(\cos \theta)$  has a scaling limit: as  $l \rightarrow \infty$ , for any *fixed*

$$\psi \in \left[0, (l + 1/2) \frac{\pi}{2}\right],$$

we have

$$(9) \quad P_l \left( \cos \left( \frac{\psi}{l + 1/2} \right) \right) \approx J_0(\psi).$$

In fact, the latter estimate holds uniformly for  $\psi = o(l)$ , and it may be shown that, as a tail of a conditionally convergent integral, the contribution of the other regime  $\psi \gg l$  is negligible (see the proof of Proposition 4.2).

In this sequel we neglect the difference between  $l$  and  $l + \frac{1}{2}$ . It is then natural to try to replace  $P_l(\cos \theta)$  in (8) with its scaling limit; a formal

substitution yields heuristically

$$\begin{aligned}
 (10) \quad I_l &\approx \frac{1}{l} \int_0^{l\pi/2} \arcsin(P_l(\cos(\psi/l))) \sin(\psi/l) d\psi \approx \frac{1}{l} \int_0^{l\pi/2} \arcsin(J_0(\psi)) \sin(\psi/l) d\psi \\
 &\approx \frac{1}{l^2} \int_0^{l\pi/2} \arcsin(J_0(\psi)) \psi d\psi,
 \end{aligned}$$

where we replaced  $\sin\left(\frac{\psi}{l}\right)$  with  $\frac{\psi}{l}$ , which is justified for  $\psi = o(l)$ . This is inconsistent to the statement of Theorem 1.2; the integral

$$(11) \quad \int_0^\infty \arcsin(J_0(\psi)) \psi d\psi,$$

diverges, so some more care is needed to transform (10). Moreover, the last integrand in (10) is somewhat different from the integrand in the integral (6) defining  $C$  (neglecting the constant in front of the integral in (6)) in that in (6) we subtract  $J_0(\psi)$  from the arcsine. Note that the factor  $\psi$  in both (10) and (6) is reminiscent of the uniform measure  $\sin \theta d\theta$  on the sphere.

There is a subtlety that explains the latter discrepancy between the integrand in the definition (6) of  $C$  and the integrand in (10). One way to justify the substitution of the scaling limit of  $P_l(\cos \theta)$  (i.e. the second step of (10)) is expanding the arcsine in the integral of (8) into the Taylor series around the origin

$$\arcsin(t) = \sum_{k=0}^{\infty} a_k t^{2k+1},$$

for some explicitly given coefficients  $a_k$  (see (19) and the formula immediately after). We then need to evaluate all the odd moments of  $P_l(\cos \theta)$  w.r.t. to the measure  $\sin \theta d\theta$  on  $[0, \frac{\pi}{2}]$ . It turns out that the moments (appropriately scaled) are asymptotically equivalent to the corresponding moments of the Bessel function on  $\mathbb{R}_+$  w.r.t. the measure  $\psi d\psi$ , with the exception for the first moment (Lemma 5.2). Indeed,  $P_l(\cos \theta)$  integrates to zero, whereas  $\int_0^\infty J_0(\psi) \psi d\psi$  diverges; this phenomenon also accounts for the divergence of (11). To account for this difference, we need to subtract  $J_0(\psi)$  from the arcsine in the integrand (10), thus obtaining a conditionally convergent integral. This is indeed the heuristic explanation for the discrepancy between (10) and (6).

To summarize this discussion, we note the following. The variance is given, up to an explicit constant, by the integral (8). The constant  $C$  in (6) is reminiscent of  $I_l$  up to the scaling (9): the expression  $P_l(\cos \theta)$  is replaced by its scaling limit  $J_0(\psi)$ , and the measure  $\sin \theta d\theta$  becomes  $\psi d\psi$  on  $\mathbb{R}_+$  after scaling. Subtracting  $J_0(\psi)$  from the integrand in (6) accounts for the fact that for even  $l$ , the integral of  $P_l(\cos \theta)$  against  $\sin \theta d\theta$  vanishes, whereas this is not the case of integral of the scaling limit  $J_0(\psi)$  against  $\psi d\psi$ .

## 3. DISCUSSION

As reported in Section 1.3, the asymptotics for the variance of the excursion sets is given for  $z \neq 0$  by

$$\text{Var}(\Phi_l(z)) \sim z^2 \phi(z)^2 \cdot \frac{4\pi^2}{l}$$

(here  $\phi$  is the Gaussian probability density); for the defect ( $z = 0$ ), this gives only an upper bound  $o(\frac{1}{l})$ . The principal result of the present paper (Theorem 1.2) states that the defect variance (5) is of order of magnitude  $l^{-2}$ . Our explanation for this discrepancy is the disappearance, for  $z = 0$ , of quadratic term in the Hermite expansion of the function  $\mathbb{1}(f_l \leq z)$ , see [12].

We should compare this situation to the length distribution of the level curves. For  $t \in \mathbb{R}$  let

$$\mathcal{L}_l^t = \mathcal{L}^t(f_l) = \text{length}(f_l^{-1}(t)).$$

be the (random) length of the level curve  $f_l^{-1}(t)$ , the most important case being that of  $t = 0$ ; the corresponding level curve is called the *nodal line*. It is known [20] that for  $t \in \mathbb{R}$ , the expected length is

$$\mathbb{E}[\mathcal{L}_l^t] \sim c_1 e^{-t^2/2} \cdot l.$$

The variance is asymptotic to

$$\text{Var}(\mathcal{L}_l^t) \sim c_2 e^{-t^2} t^4 \cdot l$$

for  $t \neq 0$ , whereas for the nodal length

$$\text{Var}(\mathcal{L}_l^0) \sim c_3 \log l.$$

It is therefore natural to conjecture that the discrepancy of the defect and the empirical measure are somehow related to the discrepancy in the variance of the nodal length. However, no simple explanation, as in the case of the defect, is known to explain the discrepancy between the nodal length and the non-vanishing level curves length.

Note that we may relate  $\mathcal{L}_l^t$  to the excursion sets by the identity

$$(12) \quad \Phi_l(z) = \int_{-\infty}^z \mathcal{L}_l^t dt.$$

Let us recall that for two random variables  $X, Y$ , the correlation is defined as

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)} \sqrt{\text{Var}(Y)}}.$$

It is known [20] that for every  $t_1, t_2 \in \mathbb{R}$ ,  $\mathcal{L}_l^t$  become asymptotically fully dependent for large  $l$ , in the sense that<sup>1</sup>

$$(13) \quad \text{Corr}(\mathcal{L}_l^{t_1}, \mathcal{L}_l^{t_2}) = 1 - o_{l \rightarrow \infty}(1).$$

In fact, a much stronger statement regarding the rate of convergence was proven. It then seems, that the asymptotic degeneracy of  $\Phi_l(z)$  is, essentially, an artifact of (12) and the asymptotic full dependence of the level lengths (13).

A possible explanation for the phenomenon (13) is the following conjecture due to Mikhail Sodin. Let  $x \in \mathcal{S}^2$  and for  $t \in \mathbb{R}$  let  $\mathcal{L}_{l;x}^t = \mathcal{L}_x^t(f_l)$  (the “local length”) be the (random) length of the unique component of  $f_l^{-1}(t)$  that contains  $x$  inside (or 0, if  $f_l$  does not cross the level  $t$ ).

**Conjecture 3.1** (M. Sodin). *The local lengths are asymptotically fully dependent in the sense that for every  $x \in \mathcal{S}^2$  and  $t_1, t_2 \in \mathbb{R}$*

$$\text{Corr}(\mathcal{L}_{l;x}^{t_1}, \mathcal{L}_{l;x}^{t_2}) = 1 - o_{l \rightarrow \infty}(1).$$

Heuristically, Conjecture 3.1 is stronger than (13) (and thus the asymptotic degeneracy of the empirical measure via (12)), since  $\mathcal{L}^t$  can be viewed as a summation of  $\mathcal{L}_x^t$  over a set of points  $x \in \mathcal{S}^2$ . More rigorously, we should assume that  $f_l(x) \neq t$ ; it is immediate to see that this is satisfied almost surely.

#### 4. PROOF OF THEOREM 1.2

The first Lemma is probably already known, but we failed to locate a direct reference and thus we report it for completeness. The proof is based on standard properties of Gaussian random variables.

**Lemma 4.1.** *For  $l$  even we have*

$$(14) \quad \text{Var}(\mathcal{D}_l) = 32\pi \int_0^{\pi/2} \arcsin(P_l(\cos \theta)) \sin \theta d\theta.$$

We postpone the proof of Lemma 4.1 until the end of the present section. Let us denote the integral in (14)

$$(15) \quad I_l = \int_0^{\pi/2} \arcsin(P_l(\cos \theta)) \sin \theta d\theta,$$

so that evaluating the defect variance is equivalent to evaluating  $I_l$ , which is done in the following proposition (for even  $l$ ):

**Proposition 4.2.** *As  $l \rightarrow \infty$  along even numbers, we have*

$$(16) \quad I_l = \frac{C_1}{l^2} + o_{l \rightarrow \infty}\left(\frac{1}{l^2}\right),$$

where

$$(17) \quad C_1 = \int_0^\infty \psi(\arcsin(J_0(\psi)) - J_0(\psi)) d\psi.$$

Moreover, the constant  $C_1$  is strictly positive.

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<sup>1</sup>The asymptotic full dependence (13) was proven under the technical assumption  $t_1, t_2 \neq 0$ ; it is natural to conjecture, though, that a slight modification of the same argument will work if either of  $t_i$  vanishes.



*Proof of Theorem 1.2 assuming Proposition 4.2.* Formula (14) together with Proposition 4.2 yields

$$\mathrm{Var}(\mathcal{D}_l) = \frac{C}{l^2} (1 + o_{l \rightarrow \infty}(1)).$$

The positivity of the constant  $C = 32\pi C_1$  follows directly from the positivity of  $C_1$ , which is claimed in Proposition 4.2.  $\square$

*Proof of Proposition 4.2.* As it was explained in Section 2, to extract the asymptotics of  $I_l$ , we will expand the arcsine on the RHS of (15) into the Taylor series around the origin; we will encounter only the odd moments of  $P_l(\cos \theta)$ , due to the arcsine being an odd function. As it was pointed out in Section 2, the function  $P_l(\cos \theta)$  differ from its scaling limit in that the integral of the former vanishes, whereas the integral of the latter diverges; all the other odd moments of  $P_l(\cos \theta)$  are asymptotic to the (properly scaled) corresponding moments of the Bessel function, (see Lemma 5.2). To account for this discrepancy, we subtract  $P_l(\cos \theta)$  from the integrand and add it back separately. To this end we write

$$\arcsin(P_l(\cos \theta)) = (\arcsin(P_l(\cos \theta)) - P_l(\cos \theta)) + P_l(\cos \theta),$$

so that the vanishing of the integral of the latter

$$\int_0^{\pi/2} P_l(\cos \theta) \sin \theta d\theta = 0,$$

for even  $l$  implies

$$(18) \quad I_l = \int_0^{\pi/2} (\arcsin(P_l(\cos \theta)) - P_l(\cos \theta)) \sin \theta d\theta.$$

The advantage of the latter representation (18) over (15) is that the only powers that will appear in the Taylor expansion of the arcsine on the right-hand side of (18) are of order  $\geq 3$ , so that the moments of  $P_l(\cos \theta)$  are all identical to the corresponding moments of the scaling limit. Intuitively, this means that we may replace the appearances of  $P_l(\cos \theta)$  in (18) by its the scaling limit. The rest of the present proof is a rigorous argument that will establish the latter statement.

Let

$$(19) \quad \arcsin(t) - t = \sum_{k=1}^{\infty} a_k t^{2k+1},$$

where

$$(20) \quad a_k = \frac{(2k)!}{4^k (k!)^2 (2k+1)}$$

are the Taylor coefficients of the arcsine. Note that all the terms in the expansion (19) are positive, and by the Stirling formula, the coefficients are asymptotic to

$$(21) \quad a_k \sim \frac{c}{k^{3/2}}$$

for some  $c > 0$ , so that, in particular, the Taylor series (19) is uniformly absolutely convergent. Therefore we may write

$$(22) \quad I_l = \sum_{k=1}^{\infty} a_k \int_0^{\pi/2} P_l(\cos \theta)^{2k+1} \sin \theta d\theta.$$

We know from Lemma 5.2, that for every  $k \geq 1$ ,

$$(23) \quad \int_0^{\pi/2} P_l(\cos \theta)^{2k+1} \sin \theta d\theta \sim \frac{c_{2k+1}}{l^2},$$

with  $c_{2k+1}$  given by (33); comparing this result to (22), it is natural to expect that

$$(24) \quad I_l = \frac{C_2}{l^2} + o\left(\frac{1}{l^2}\right),$$

where

$$(25) \quad C_2 = \sum_{k=1}^{\infty} a_k c_{2k+1}.$$

We are going to formally prove (24) immediately; however, first we evaluate the constant  $C_2$  by summing up the series in (25), and validate that indeed  $C_2 = C_1$  in (17), as claimed. We plug (33) into (25) to formally compute

$$\begin{aligned} C_2 &= \sum_{k=1}^{\infty} a_k \int_0^{\infty} \psi J_0(\psi)^{2k+1} d\psi = \int_0^{\infty} \psi \cdot \left( \sum_{k=1}^{\infty} a_k J_0(\psi)^{2k+1} \right) d\psi \\ &= \int_0^{\infty} \psi \cdot (\arcsin(J_0(\psi)) - J_0(\psi)) d\psi = C_1, \end{aligned}$$

where to obtain the third equality, we used (19) again. To justify the exchange of the summation and integration order, we consider the finite summation

$$\sum_{k=1}^m a_k \int_0^{\infty} \psi J_0(\psi)^{2k+1} d\psi,$$

using (21) and (31) to bound the contribution of tails, and take the limit  $m \rightarrow \infty$ .

We now turn to prove (24). To this end we expand (19) into a *finite* degree Taylor polynomial while controlling the tail using Lemma 5.7. Indeed, using (21) and (46), we easily obtain

$$\begin{aligned} \sum_{k=m+1}^{\infty} a_k \int_0^{\pi/2} |P_l(\cos \theta)|^{2k+1} \sin \theta d\theta &\leq \sum_{k=m+1}^{\infty} a_k \int_0^{\pi/2} |P_l(\cos \theta)|^5 \sin \theta d\theta \\ &\ll \frac{1}{l^2} \cdot \sum_{k=m+1}^{\infty} \frac{1}{k^{3/2}} \ll \frac{1}{\sqrt{ml^2}}, \end{aligned}$$

since  $|P_l(t)| \leq 1$  for every  $l$  and  $t \in [-1, 1]$ , so that  $|P_l(t)|^k$  is monotone decreasing with  $k$ .

We then have for every  $m$  ( $m = m(l)$  to be chosen)

$$(26) \quad I_l = \sum_{k=1}^m a_k \int_0^{\pi/2} P_l(\cos \theta)^{2k+1} \sin \theta d\theta + O\left(\frac{1}{\sqrt{ml^2}}\right),$$

and plugging (23) (a direct consequence of Lemma 5.2) into (26) finally yields

$$(27) \quad I_l = C_{2,m} \cdot \frac{1}{l^2} + o_m\left(\frac{1}{l^2}\right) + \frac{1}{\sqrt{ml^2}}$$

with

$$C_{2,m} = \sum_{k=1}^m a_k c_{2k+1}.$$

It is clear that (27) implies (24) (recall the definition (25) of  $C_2$  and note that as  $m \rightarrow \infty$ ,  $C_{2,m} \rightarrow C_2$ ), which concludes the proof of the statement (16) of the present proposition.

It then remains to prove the positivity of the constant  $C_1$ . While its nonnegativity  $C_1 \geq 0$  is clear since it is the leading constant for the integral  $I_1$ , which, up to an explicit positive constant, equals the variance of a random variable via (14), the strict positivity is less obvious. To this end, we recall that  $C_1 = C_2$ , the latter being given by (25). Note that all the Taylor coefficients  $a_k$  of arcsine are positive (see (20)). Hence the positivity of  $C_2$  (and thus also, of  $C_1$ ) follows from the second statement of Lemma 5.2, which claims that  $c_k \geq 0$  are nonnegative, and  $c_3 > 0$  is explicitly given.  $\square$

*Proof of Lemma 4.1.* The result of the present lemma is an artifact of the following general fact (see e.g. [14, 15]). Let  $(X_1, X_2)$  be a 2-variate centred Gaussian random variable with covariance matrix

$$\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix},$$

$|r| \leq 1$ , and for  $i = 1, 2$  define the random variables (recall that  $H(t)$  is the Heaviside function (4))

$$B_i = H(X_i) = \begin{cases} 1 & X_i > 0 \\ -1 & X_i < 0 \end{cases}.$$

Then  $B_i$  are both mean zero with covariance given by

$$\text{Cov}(B_1, B_2) = \frac{2}{\pi} \arcsin(r).$$

Using the definition (3) of the defect, we may exchange the order of taking the expectation and integrating to write

$$(28) \quad \begin{aligned} \text{Var}(\mathcal{D}_l) &= \mathbb{E}[\mathcal{D}_l^2] = \mathbb{E} \iint_{\mathcal{S}^2 \times \mathcal{S}^2} H(f_l(x)) H(f_l(y)) dx dy \\ &= \iint_{\mathcal{S}^2 \times \mathcal{S}^2} \mathbb{E}[H(f_l(x)) H(f_l(y))] dx dy = 4\pi \int_{\mathcal{S}^2} \mathbb{E}[H(f_l(N)) H(f_l(x))] dx, \end{aligned}$$

by the isotropic property of the random field  $f_l$ , where  $N$  is the northern pole. Note that for every  $x \in \mathcal{S}^2$ ,  $f_l(N)$  and  $f_l(x)$  are jointly Gaussian, centred, with unit variance and covariance equal to

$$\mathbb{E}[f_l(N) \cdot f_l(x)] = r_l(N, x) = P_l(\cos \theta),$$

where  $(\theta, \phi)$  are the spherical coordinates of  $x$ ; the latter follows from the definition (2) of the covariance function. Therefore, as it was explained earlier, for every  $x, y \in \mathcal{S}^2$ ,

$$\mathbb{E}[H(f_l(x)) H(f_l(y))] = \frac{2}{\pi} \arcsin(P_l(\cos \theta)).$$

We then evaluate the latter integral in (28) in the spherical coordinates as

$$\text{Var}(\mathcal{D}_l) = 8\pi^2 \int_0^\pi \frac{2}{\pi} \arcsin(P_l(\cos \theta)) \sin \theta d\theta = 16\pi \int_0^\pi \arcsin(P_l(\cos \theta)) \sin \theta d\theta,$$

which, taking into account  $l$  being even (and thus  $P_l(t)$  is also even), is the statement (14) of the present lemma.  $\square$

## 5. MOMENTS OF LEGENDRE POLYNOMIALS

We start by recalling a basic fact on the asymptotic behaviour of Legendre polynomials (see for instance [16]); as usual, we shall denote by  $J_\nu$  the Bessel function of the first kind.

**Lemma 5.1** (Hilb's asymptotics). *For any  $\epsilon > 0$ ,  $C > 0$  we have*

$$(29) \quad P_l(\cos \theta) = \left( \frac{\theta}{\sin(\theta)} \right)^{1/2} J_0((l + 1/2)\theta) + \delta(\theta),$$

where

$$(30) \quad \delta(\theta) \ll \begin{cases} \theta^{1/2} l^{-3/2} & \theta > \frac{C}{l} \\ \theta^2 & 0 < \theta < \frac{C}{l} \end{cases}.$$

uniformly w.r.t.  $l \geq 1$ ,  $\theta \in [0, \pi - \epsilon]$ .

Note that we will use Lemma 5.1 only for the range  $\theta \in [0, \frac{\pi}{2}]$ , so that we may forget about the  $\epsilon$  altogether. We will also recall that (see again [16])

$$(31) \quad |J_0(\psi)| = O\left(\frac{1}{\sqrt{\psi}}\right).$$

**Lemma 5.2.** *Let  $j \geq 5$  or  $j = 3$ . Then*

$$(32) \quad \int_0^{\pi/2} P_l(\cos \theta)^j \sin \theta d\theta = c_j \frac{1}{l^2} (1 + o_j(1)),$$

where the constants  $c_j$  are given by

$$(33) \quad c_j = \int_0^\infty \psi J_0(\psi)^j d\psi,$$

the RHS of (33) being absolutely convergent for  $j \geq 5$  and conditionally convergent for  $j = 3$ . Moreover, for every  $j$  as above, the constants  $c_j \geq 0$  are nonnegative, and  $c_3 > 0$  is positive, given explicitly by

$$(34) \quad c_3 = \frac{2}{\pi\sqrt{3}}.$$

*Proof.* By the Hilb's asymptotics, we have

$$(35) \quad \int_0^{\pi/2} P_l(\cos \theta)^j \sin \theta d\theta = \int_0^{\pi/2} \left( \left( \frac{\theta}{\sin(\theta)} \right)^{1/2} J_0((l + 1/2)\theta) + \delta(\theta) \right)^j \sin \theta d\theta.$$

The contribution of the error term to (35) is (exploiting  $\frac{\theta}{\sin \theta}$  being bounded)

$$(36) \quad \ll \int_0^{\pi/2} |J_0((l + 1/2)\theta)|^{j-1} \delta(\theta) \theta d\theta = \int_0^{1/l} + \int_{1/l}^{\pi/2}.$$

Now

$$\int_0^{1/l} |J_0((l + 1/2)\theta)|^{j-1} \delta(\theta) \theta d\theta \ll \int_0^{1/l} \theta^3 d\theta \ll \frac{1}{l^4},$$

and using (31), we may bound the second integral in (36) as

$$\int_{1/l}^{\pi/2} |J_0((l + 1/2)\theta)|^{j-1} \delta(\theta) \theta d\theta \ll \frac{1}{l^{(j+2)/2}} \int_{1/l}^{\pi/2} \theta^{3/2} d\theta \ll \frac{1}{l^{(j+2)/2}}.$$

Plugging the last couple of estimates into (36), and finally into (35), we obtain for  $j \geq 3$

$$(37) \quad \int_0^{\pi/2} P_l(\cos \theta)^j \sin \theta d\theta = \int_0^{\pi/2} \left( \frac{\theta}{\sin(\theta)} \right)^{j/2-1} J_0((l + 1/2)\theta)^j \theta d\theta + O\left(\frac{1}{l^{5/2}}\right).$$

Therefore we are to evaluate

$$\int_0^{\pi/2} \left( \frac{\theta}{\sin(\theta)} \right)^{j/2-1} J_0((l+1/2)\theta)^j \theta d\theta = \frac{1}{L^2} \int_0^{L\pi/2} \left( \frac{\psi/L}{\sin(\psi/L)} \right)^{j/2-1} J_0(\psi)^j \psi d\psi,$$

where we denote  $L := l + \frac{1}{2}$  for brevity. The statement of the present lemma is then equivalent to

$$(38) \quad \int_0^{L\pi/2} \left( \frac{\psi/L}{\sin(\psi/L)} \right)^{j/2-1} J_0(\psi)^j \psi d\psi \rightarrow c_j,$$

where  $c_j$  is defined by (33).

The main idea is that the leading contribution is provided from the intermediate range  $1 \ll \psi \ll \epsilon L$  for any  $\epsilon > 0$ ; here we may replace the factor

$$\left( \frac{\psi/L}{\sin(\psi/L)} \right)^{j/2-1}$$

with 1. To make this argument precise, we write for  $\psi \in [0, \frac{\pi}{2} \cdot L]$

$$\frac{\psi/L}{\sin(\psi/L)} = 1 + O\left(\frac{\psi^2}{L^2}\right),$$

so that also

$$(39) \quad \left( \frac{\psi/L}{\sin(\psi/L)} \right)^{j/2-1} = 1 + O_j\left(\frac{\psi^2}{L^2}\right).$$

Therefore the integral in (38) is

$$(40) \quad \int_0^{L\pi/2} \left( \frac{\psi/L}{\sin(\psi/L)} \right)^{j/2-1} J_0(\psi)^j \psi d\psi = \int_0^{L\pi/2} J_0(\psi)^j \psi d\psi + O\left(\frac{1}{L^2} \int_0^{L\pi/2} \psi^3 |J_0(\psi)|^j d\psi\right).$$

Note that as  $l \rightarrow \infty$  (equivalently  $L \rightarrow \infty$ ), the main term on the RHS of converges to  $c_j$ ; indeed

$$(41) \quad \int_0^{L\pi/2} J_0(\psi)^j \psi d\psi \rightarrow c_j,$$

so that it remains to bound the error term. Now

$$(42) \quad \int_0^{L\pi/2} \psi^3 |J_0(\psi)^j| d\psi = \int_0^1 + \int_1^{L\pi/2} = O(1) + \int_1^{L\pi/2},$$

and we use (31) to bound the latter as

$$\int_1^{L\pi/2} \psi^3 |J_0(\psi)^j| d\psi \ll \int_1^{L\pi/2} \psi^{3-j/2} d\psi = O(1 + l^{4-j/2}),$$

so that upon plugging the latter into (42) yields

$$\int_0^{L\pi/2} \psi^3 |J_0(\psi)^j| d\psi = O(1 + l^{4-j/2});$$

plugging the latter into (40) yields

$$\int_0^{L\pi/2} \left( \frac{\psi/L}{\sin(\psi/L)} \right)^{j/2-1} J_0(\psi)^j \psi d\psi = \int_0^{L\pi/2} J_0(\psi)^j \psi d\psi + O(l^{-2} + l^{2-j/2}),$$

so that (41) implies (38) for  $j \geq 5$ , which was equivalent to the statement of the present lemma in this case.

It then remains to prove the result for  $j = 3$ , for which case we have to work a little harder due to the conditional convergence of the integral (33); to treat this technicality we will have to exploit the oscillatory behaviour of the Bessel function, and not only its decay (31). It is well-known that

$$J_0(\psi) = \sqrt{\frac{2}{\pi}} \frac{\cos(\psi - \pi/4)}{\sqrt{\psi}} + O\left(\frac{1}{\psi^{3/2}}\right),$$

so that

$$(43) \quad \left(\frac{2}{\pi}\right)^{3/2} \frac{\cos(\psi - \pi/4)^3}{\psi^{3/2}} + O\left(\frac{1}{\psi^{5/2}}\right),$$

and hence the integral on the RHS (33) is indeed convergent, by integration by parts.

Now we choose a (large) parameter  $K \gg 1$ , divide the integration range into  $[0, K]$  and  $[K, L\pi/2]$ ; the main contribution comes from the first term, whence we need to prove that the latter vanishes. Indeed, we use (43) to bound

$$(44) \quad \int_K^{L\pi/2} \left( \frac{\psi/L}{\sin(\psi/L)} \right)^{1/2} J_0(\psi)^3 \psi d\psi \ll \frac{1}{\sqrt{K}},$$

where we use integration by parts with the bounded function

$$I(T) = \int_0^T \cos(t)^3 dt$$

to bound the contribution leading term in (43); it is easy to bound the contribution of the error term in (43) using the crude estimate (31).

On  $[0, K]$  we use (39) to write

$$(45) \quad \int_0^K \left( \frac{\psi/L}{\sin(\psi/L)} \right)^{1/2} J_0(\psi)^3 \psi d\psi = \int_0^K J_0(\psi)^3 \psi d\psi + O\left(\frac{K^{5/2}}{l^2}\right),$$

the former clearly being convergent to  $c_3$ . Combining (44) with (45) we obtain

$$\int_K^{L\pi/2} \left( \frac{\psi/L}{\sin(\psi/L)} \right)^{1/2} J_0(\psi)^3 \psi d\psi = \int_0^K J_0(\psi)^3 \psi d\psi + O\left(\frac{1}{\sqrt{K}} + \frac{K^{5/2}}{l^2}\right);$$

this implies (38) for  $j = 3$  (which is equivalent to the statement of the present lemma for  $j = 3$ ) upon choosing the parameter  $K$  growing to infinity sufficiently slowly (e.g.  $K = \sqrt{l}$ ). Since  $j = 3$  was the only case that was not covered earlier, this concludes proof of the moments part of Lemma 5.2.

It then remains to prove the nonnegativity statement of  $c_j$ , and the explicit expression (34) for  $c_3$ . In fact, both of those statements follow from the computation we performed in our previous paper [12], p. 18. We report the relevant results in Lemma 5.3 below.  $\square$

We recall an alternative characterization for moments of Legendre polynomials from [12]. For given positive integers  $l_1, l_2, l_3$  we introduce the so-called Clebsch-Gordan coefficients  $\left\{C_{l_1 l_2 l_3}^{l_3 0}\right\}$ , which are different from zero if and only if  $l_1, l_2, l_3$  are such that  $l_1 + l_2 + l_3$  is even and  $l_i + l_j \leq l_k$  for all permutations  $i, j, k = 1, 2, 3$ . The Clebsch-Gordan coefficients are well-known in group representation theory (they intertwine alternative representations for  $SO(3)$ ) and in the quantum theory of angular momentum; we do not provide more details here, but we refer instead to [12] or to standard references such as [17, 18] (see also [10]). The results provided in [12] are as follows:

**Lemma 5.3** ([12], Lemma A.1 and above). *For all even  $l$ , we have*

$$\int_0^1 P_l(t)^3 dt = \frac{1}{2l+1} \left\{C_{l0l0}^{l0}\right\}^2,$$

and

$$\lim_{l \rightarrow \infty} \frac{1}{2l+1} \left\{C_{l0l0}^{l0}\right\}^2 = \frac{2}{\pi\sqrt{3}}.$$

Also, for  $j \geq 5$

$$\int_0^1 P_l(t)^j dt = \frac{1}{2l+1} \sum_{L_1 \dots L_{j-3}} \left\{C_{l0l0}^{L_1 0} C_{L_1 0 l 0}^{L_2 0} \dots C_{L_{j-3} 0 l 0}^{l 0}\right\}^2 > 0.$$

**Remark 5.4.** *The previous discussion yields the following interesting corollary: as  $l \rightarrow \infty$*

$$\lim_{l \rightarrow \infty} \frac{1}{2l+1} \left\{C_{l0l0}^{l0}\right\}^2 = \int_0^\infty \psi J_0(\psi)^3 d\psi,$$

and for  $j \geq 5$

$$\lim_{l \rightarrow \infty} \left[ \frac{1}{2l+1} \sum_{L_1 \dots L_{j-3}} \left\{C_{l0l0}^{L_1 0} C_{L_1 0 l 0}^{L_2 0} \dots C_{L_{j-3} 0 l 0}^{l 0}\right\}^2 \right] = \int_0^\infty \psi J_0(\psi)^j d\psi.$$

The following lemma establishes the lower bound for the constant  $C$  in Theorem 1.2 claimed in Section 1.2. Its proof is straightforward.

**Lemma 5.5.** *For  $C$  as in (5), we have*

$$C > \frac{32}{\sqrt{27}}.$$



*Proof.* Recall that  $C = \frac{2}{\pi}C_1$ , where  $C_1 = C_2$  is given by (25). Since all the  $a_j$  and  $c_k$  are nonnegative, Lemma 5.5 follows from bounding the first term in the series

$$C \geq 32\pi \times a_1 c_3 = \frac{32}{\sqrt{27}},$$

where we used  $a_1 = 1/6$  and (34).  $\square$

**Remark 5.6.** As a final remark, we note that the moments of  $P_l(\cos \theta)$  are themselves of some physical interest. In particular, while analyzing the relationship between asymptotic Gaussianity and ergodicity of isotropic spherical random fields, [11] considered the random field

$$T(x) := \sum_{l=1}^{\infty} T_l(x) = \sum_{l=1}^{\infty} c_l Y_{l0}(g \cdot x),$$

where  $Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta)$  denotes standard spherical harmonics (for  $m = 0$ ) and  $g \in SO(3)$  is a uniformly distributed random rotation in  $\mathbb{R}^3$ . It is readily seen that the resulting field is isotropic, and each of its Fourier components at frequency  $l$  is marginally distributed as (for any  $x \in \mathcal{S}^2$ )

$$Y_l = c_l \sqrt{\frac{2l+1}{4\pi}} P_l(t), \quad t \sim U[0, 1].$$

Up to normalization constants, then, the moments of the marginal law are exactly those of the Legendre polynomials we established earlier in the paper. More precisely, if we focus (as in [11]) on  $\tilde{T}_l := T_l(x)/\sqrt{\text{Var}(T_l)}$ , we obtain immediately, for any  $x \in \mathcal{S}^2$

$$\mathbb{E} \left\{ \tilde{T}_l(x) \right\} = \mathbb{E} \left\{ \sqrt{(2l+1)} P_l(t) \right\} = 0, \quad \mathbb{E} \left\{ \tilde{T}_l(x) \right\}^2 = 1,$$

and

$$\mathbb{E} \left\{ \tilde{T}_l(x) \right\}^3 = E \left\{ \sqrt{(2l+1)} P_l(t) \right\}^3 = O \left( \frac{1}{\sqrt{l}} \right),$$

whereas [19]

$$E \left\{ \sqrt{(2l+1)} P_l(t) \right\}^4 \simeq \log l,$$

and hence all moments of order  $q \geq 4$  diverge.

Note that in Lemma 5.2 we worked relatively hard to establish the precise asymptotics for the moments of Legendre polynomials. A much cruder version of the same argument gives a uniform upper bound for the 5th moment of the *absolute value* of the Legendre polynomials:

**Lemma 5.7.** *We have the following uniform upper bound for the 5th moment of the absolute value of Legendre polynomials*

$$(46) \quad \int_0^{\pi/2} |P_l(\cos \theta)|^5 \sin \theta d\theta = O \left( \frac{1}{l^2} \right),$$

where the constant involved in the ‘ $O$ ’-notation is universal.

*Proof.* The proof follows along the same lines as the beginning of proof of Lemma 5.2 for  $k \geq 5$ , except that we use the crude upper bound (31) whenever we reach (37), and the trivial inequality

$$\frac{\theta}{\sin \theta} \ll 1, \text{ for } \theta \in \left[0, \frac{\pi}{2}\right].$$

□

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